QUASIDETERMINANTS AND q-COMMUTING MINORS

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Abstract. We present two new proofs of the the important q-commuting property holding among certain pairs of quantum minors of an $n \times n$ q-generic matrix. The first uses elementary quasideterminantal arithmetic; the second involves paths in an edge-weighted directed graph.

1. Introduction & Main Theorem

This paper arose from an attempt to understand the "quantum shape algebra" of Taft and Towber [8], which we call the *flag algebra* $\mathcal{F}\ell_q(n)$ here. One goal was to find quasideterminantal justifications for the relations chosen for $\mathcal{F}\ell_q(n)$. A second goal was to find some hidden relations, within $\mathcal{F}\ell_q(n)$, known to hold in an isomorphic image. To more quickly reach a statement of the theorem, we save further remarks on the goals for later.

Definition 1. Given two subsets $I, J \subseteq [n]$, we say J *surrounds*[1] I, written $J \curvearrowright I$, if (i) $|J| \leq |I|$, and (ii) there exist disjoint subsets $\emptyset \subseteq J', J'' \subseteq J$ such that:

a. $J \setminus I = J' \dot\cup J''$,
b. $j' < i$ for all $j' \in J'$ and $i \in I \setminus J$,
c. $i < j''$ for all $i \in I \setminus J$ and $j'' \in J'$,

In this case, we put $\langle\!\langle J, I\rangle\!\rangle = |J''| - |J'|$.

Given an $n \times n$ *q-generic matrix* X and a subset $I \subseteq [n]$ with $|I| = d$, we write $[\![I]\!]$ for the quantum minor built from X by taking row-set I and column-set $[d]$.

Theorem 1 (q-Commuting Minors). *If the subsets $I, J \subseteq [n]$ satisfy $J \curvearrowright I$, the quantum minors $[\![J]\!]$ and $[\![I]\!]$ q-commute. Specifically,*

$$[\![J]\!][\![I]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I]\!][\![J]\!]\,. \tag{1}$$

An earlier proof of this theorem may be found in [5], while Leclerc and Zelevinsky [7] show that if $[\![J]\!][\![I]\!] = q^{\alpha}[\![I]\!][\![J]\!]$ for some $\alpha \in \mathbb{Z}$, then $J \curvearrowright I$. We give two new proofs in the sequel. The first proof ($\mathcal{Q}$) uses simple arithmetic involving quasideterminants; the second ($\mathcal{G}$) involves counting weighted paths on a directed graph.

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[1] In the literature, sets J and I sharing this relationship are called "weakly separated." We avoid this terminology because it does not indicate who separates whom.

1.1. **Useful notation.** The reader has already encountered our notation $[n]$ for the set $\{1, 2, \ldots, n\}$; let $\binom{[n]}{d}$ denote the set of all subsets of $[n]$ of size d. Given a set $I = \{i_1 < i_2 < \cdots i_d\} \in \binom{[n]}{d}$ and any $I' \subseteq I$, we write $I^{I'}$ for the subset built from I by *deleting* I' (i.e. $I \setminus I'$) and $I_{I'}$ for the complement (i.e. a fancy way of saying *keep* I'). In case $\Lambda = \{\lambda_1 < \lambda_2 \cdots < \lambda_r\} \in \binom{[d]}{r}$, we write $I_{(\Lambda)}$ for the subset $\{i_{\lambda_1}, i_{\lambda_2}, \ldots, i_{\lambda_r}\}$ and $I^{(\Lambda)}$ for the complement.

Suppose instead $I \in [n]^d$, the set of all d-tuples chosen from $[n]$. In this case, the notations $I_{I'}$ and $I^{I'}$ are not well-defined (as the entries of I' may occur in more than one place within I) but the notations $I_{(\Lambda)}$ and $I^{(\Lambda)}$ will be useful in the sequel. If I, J are two sets or tuples of sizes d, e respectively, we define $A|B$ to be the $(d+e)$-tuple $(i_1, \ldots, i_d, j_1, \ldots, j_e)$. Let $[n]^d_* \subseteq [n]^d$ denote those d-tuples with distinct entries. For $I \in [n]^d_*$, we define the *length* of I to be $\ell(I) = \#\mathrm{inv}(I) = \#\big\{(j,k) : j < k \text{ and } i_j > i_k\big\}$. Fix $i \in [n]$ and $I = i_1, i_2, \ldots, i_d$ (viewed either as a set or a d-tuple without repetition); if there is a $1 \leq k \leq d$ with $i_k = i$, then k is the *position* of i and we write $\mathrm{pos}_I(i) = k$.

We extend our delete/keep notation to matrices. Let A be an $n \times n$ matrix whose rows and columns are indexed by R and C, respectively. For any $R' \subseteq R$ and $C' \subseteq C$, we let $A^{R',C'}$ denote the submatrix built from A by deleting row-indices R' and column-indices C'. Let $A_{R',C'}$ be the complementary submatrix. In case $R' = \{r\}$ and $C' = \{c\}$, we may abuse notation and write, e.g., A^{rc}. We will also need a means to construct matrices from A whose rows (columns) are repeated or are not in their natural order. If $I \in R^d$ and $J \in C^e$, let $A_{I,J}$ denote the obvious new matrix built from A.

2. Preliminaries for $\mathcal{Q}$-Proof

2.1. **Quasideterminants.** The quasideterminant [1, 3] was introduced by Gelfand and Retakh as a replacement for the determinant over noncommutative rings $\mathcal{R}$. Given an $n \times n$ matrix $A = (a_{ij})$ over $\mathcal{R}$, the quasideterminant $|A|_{ij}$ (there is one for each position (i,j) in the matrix) is not polynomial in the entries a_{ij} but rather a rational expression, as we will soon see. Consequently, quasideterminants are not always defined. Below is a sufficient condition (cf. loc. cit. for more details).

Definition 2. Given A and $\mathcal{R}$ as above, if A^{ij} is invertible over $\mathcal{R}$, then the (ij)-*quasideterminant* is defined and given by

$$|A|_{ij} = a_{ij} - \rho_i \cdot (A^{ij})^{-1} \cdot \chi_j\,,$$

where ρ_i is the i-th row of A with column j deleted and χ_j is the j-th column of A with row i deleted.

Remark 1. One deduces that $|A|_{ij}^{-1} = (A^{-1})_{ji}$ when both sides are defined.

Details on this remark and the following three theorems may be found in [3, 5, 6]. Note that the phrase 'when defined' is implicit throughout.

Theorem 2 (Homological Relations)**.** *Let A be a square matrix and let $i \neq j$ ($k \neq l$) be two row (column) indices. We have*

$$-|A^{jk}|_{il}^{-1} \cdot |A|_{ik} = |A^{ik}|_{jl}^{-1} \cdot |A|_{jk}.$$

Theorem 3 (Muir's Law of Extensible Minors)**.** *Let $A = A_{R,C}$ be a square matrix with row (column) indices R (C). Fix $R_0 \subsetneq R$ and $C_0 \subsetneq C$. Say an algebraic, rational expression $\mathcal{I} = \mathcal{I}(A, R_0, C_0)$ involving the quasi-minors $\{|A_{R',C'}|_{rc} : r \in R' \subseteq R_0\,,\ c \in C' \subseteq C_0\}$ is an* identity *if the equation $\mathcal{I} = 0$ is valid. For any $L \subseteq R \setminus R_0$ and $M \subseteq C \setminus C_0$, the expression $\mathcal{I}'$ built from $\mathcal{I}$ by extending all minors $|A_{R',C'}|_{rc}$ to $|A_{L\cup R',M\cup C'}|_{rc}$ is also an identity.*

Definition 3. Let B be an $n \times d$ matrix. For any $i, j, k \in [n]$ and $M \subseteq [n] \setminus \{i\}$ ($|M| = d - 1$), define $r_{ji}^{M} = r_{ji}^{M}(B) := |B_{(j|M),[d]}|_{jk}|B_{(i|M),[d]}|_{ik}^{-1}$. Gelfand and Retakh [2] show this ratio is independent of k, and call it a *right-quasi-Plücker* coordinate for B.

Remark 2. In case B is $n \times m$ for some $m > d$, we choose the first d columns of B to form the above ratio unless otherwise indicated.

Theorem 4 (Quasi-Plücker Relations)**.** *Fix an $n \times n$ matrix A, subsets $M, L \subseteq [n]$ with $|M| + 1 \leq |L|$, and $i \in [n] \setminus M$. We have the* quasi-Plücker relation ($\mathcal{P}_{L,M,i}$)

$$1 = \sum_{j \in L} r_{ij}^{L\setminus j} r_{ji}^{M}\,.$$

2.2. **Quantum determinants.** An $n \times n$ matrix $X = (x_{ab})$ is said to be q-generic if its entries satisfy the relations

$$\begin{array}{rrcl}
(\forall i,\, \forall k < l) & x_{il}x_{ik} & = & q x_{ik}x_{il} \\
(\forall i < j,\, \forall k) & x_{jk}x_{ik} & = & q x_{ik}x_{jk} \\
(\forall i < j,\, \forall k < l) & x_{jk}x_{il} & = & x_{il}x_{jk} \\
(\forall i < j,\, \forall k < l) & x_{jl}x_{ik} & = & x_{ik}x_{jl} + (q - q^{-1})x_{il}x_{jk}\,.
\end{array}$$

Notice that every submatrix of a q-generic matrix is again q-generic.

Fix a field $\Bbbk$ of characteristic 0 and a distinquished invertible element $q \in \Bbbk$ not equal to a root of unity. Let $\mathrm{M}_q(n)$ be the $\Bbbk$-algebra with n^2 generators x_{ab} subject to the relations making X a q-generic matrix. It is known [4] that $\mathrm{M}_q(n)$ is a (left) Ore domain with (left) field of fractions $D_q(n)$.

Definition 4. Given any $d \times d$ matrix A, define $\det_q A$ by

$$\det_q A = \sum_{\sigma \in \mathfrak{S}_d} (-q)^{-\ell(\sigma)} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(d),d}\,.$$

When $A = X_{R,C}$ is a submatrix of X, we have: (i) this quantity agrees with the analogous quantity modeled after the column-permutation definition of the determinant, (ii) swapping two adjacent rows of A introduces a

q^{-1}, and (iii) allowing any row of A to appear twice yields zero. Properties (ii) and (iii) allow us to uniquely define the determinant of $A = X_{I,C}$ for any $I \in [n]^d$ and $C \in \binom{[n]}{d}$. In case $C = \{1, 2 \ldots, d\}$, we introduce the shorthand notation $\det_q A = [\![I]\!]$. We will also need the case $C = s + [d] := \{s+1, s+2, \ldots, s+d\}$ for some $s > 0$, which we write as $[\![I; s]\!]$.

Properties (i)–(iii) give us the important

Theorem 5 (Quantum Determinantal Identities)**.** *Let $A = X_{R,C}$ be a $d \times d$ submatrix of X. Then for all $i, j \in R$ and $k \in C$, we have:*

$$\sum_{c \in C} A_{jc} \cdot \left\{ (-q)^{\mathrm{pos}_I(i) - \mathrm{pos}_C(c)} \mathrm{det}_q A^{ic} \right\} = \delta_{ij} \cdot \mathrm{det}_q A$$

$$\Big[\mathrm{det}_q A \ , \ A_{ik} \Big] = 0 \,.$$

In particular every submatrix of X is invertible in $D_q(n)$ and (after Remark 1) we are free to use the preceding quasideterminantal formulas on matrices built from X. The important formula follows: for all $I \in [n]^d$

$$|X_{I,\{s+1,\ldots,s+d\}}|_{i,s+d} = (-q)^{d - \mathrm{pos}_I(i)} [\![I; s]\!] \cdot [\![I^i; s]\!]^{-1} \,, \tag{2}$$

where the factors on the right commute. Theorems 2 and 5 are combined with (2) in [6] to prove

Theorem 6. *Given any $i, j \in [n]$, $\{j\} \curvearrowright \{i\}$. For any $M \subseteq [n]$, the quantum minors $[\![j|M]\!]$ and $[\![i|M]\!]$ q-commute according to equation (1).*

3. $\mathcal{Q}$-Proof of Theorem

Our first proof of Theorem 1 proceeds by induction on $|J|$ and rests on two key lemmas.

Lemma 1. If $I \subseteq [n]$ and $j \in [n] \setminus I$ satisfy $\{j\} \curvearrowright I$. Then $[\![j]\!][\![I]\!] = q^{\langle\!\langle j, I \rangle\!\rangle} [\![I]\!][\![j]\!]$.

Proof. From $(\mathcal{P}_{I,\emptyset,j})$ and (2) we have

$$1 = \sum_{i \in I} [\![j|I \setminus i]\!][\![i|I \setminus i]\!]^{-1} [\![i]\!][\![j]\!]^{-1} \,,$$

or

$$[\![j]\!] = \sum_{i \in I} [\![j|I^i]\!][\![i|I^i]\!]^{-1} [\![i]\!] \,. \tag{3}$$

Theorem 6 tells us that $[\![j|I^i]\!]$ and $[\![i|I^i]\!]$ q-commute, so we may clear the denominator in (3) on the left and get

$$[\![I]\!][\![j]\!] = \sum_{i \in I} (-q)^{\ell(i|I^i)} q^{-\langle\!\langle j, I \rangle\!\rangle} [\![j|I^i]\!][\![i]\!] \,. \tag{4}$$

In the other direction, Theorem 5 tells us that $[\![i|I^i]\!]$ and $[\![i]\!]$ commute; clearing (3) on the right yields

$$[\![j]\!][\![I]\!] = \sum_{i\in I} (-q)^{\ell(i|I^i)} [\![j|I^i]\!][\![i]\!]\,. \tag{5}$$

Compare (4) and (5) to conclude that $[\![j]\!]$ and $[\![I]\!]$ q-commute as desired. $\square$

Lemma 2. Fix $J, I \subseteq [n]$ satisfying $J \curvearrowright I$. For all $M \subseteq [n] \setminus (I \cup J)$, one has $J \cup M \curvearrowright I \cup M$ and $[\![J \cup M]\!][\![I \cup M]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I \cup M]\!][\![J \cup M]\!]$.

Proof. The first statement is clear from the definition of 'surrounds.' The second statement is a consequence of Muir's Law.

Let $J = \{j_1, \ldots, j_d\}$, $I = \{i_1, \ldots, i_e\}$, and $M = \{m_1, \ldots, m_s\}$. Because of the nature of the defining relations for q-generic matrices and the definition of quantum determinant, the expression $[\![J]\!][\![I]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I]\!][\![J]\!]$ is equivalent to $[\![J;s]\!][\![I;s]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I;s]\!][\![J;s]\!]$, or even $[\![I;s]\!]^{-1}[\![J;s]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![J;s]\!][\![I;s]\!]^{-1}$.

Let us write the left-hand side of this last equation in terms of quasideterminants:

$$\begin{aligned}
[\![I;s]\!]^{-1}[\![J;s]\!] &= \Big(|X_{I,s+[e]}|^{-1}_{i_e,s+e}\,[\![I^{(e)};s]\!]^{-1}\Big) \times \Big([\![J^{(d)};s]\!]\,|X_{J,s+[d]}|_{j_d,s+d}\Big)\\
&\;\;\vdots\\
&= |X_{I,s+[e]}|^{-1}_{i_e,s+e} \cdots |X_{(i_1,i_2),\{s+1,s+2\}}|^{-1}_{i_2,s+2}\,|X_{i_1,s+1}|^{-1}_{i_1,s+1} \;\times\\
&\quad\; |X_{j_1,s+1}|_{j_1,s+1}\,|X_{(j_1,j_2),\{s+1,s+2\}}|_{j_2,s+2} \cdots |X_{J,s+[d]}|_{j_d,s+d}.
\end{aligned}$$

Do the same to the right-hand side and get an identity involving quasideterminants. Notice that the submatrix $X_{M,[s]}$ appears nowhere in that identity. Inserting this everywhere according to Muir's Law and multiplying and dividing by $[\![M]\!]$ we get (for the left-hand side)

$$\begin{aligned}
&|X_{(I|M),[s+e]}|^{-1}_{i_e,s+e} \cdots |X_{(i_1|M),[s+1]}|^{-1}_{i_1,s+1}\,[\![M]\!]^{-1} \;\times\\
&\qquad [\![M]\!]\,|X_{(j_1|M),[s+1]}|_{j_1,s+1} \cdots |X_{(J|M),[s+d]}|_{j_d,s+d}.
\end{aligned}$$

Writing things in terms of quantum determinants again, we deduce

$$[\![I|M]\!]^{-1}[\![J|M]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![J|M]\!][\![I|M]\!]^{-1}\,.$$

Finally, note that $[\![J|M]\!][\![I|M]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I|M]\!][\![J|M]\!]$ if and only if $[\![J \cup M]\!][\![I \cup M]\!] = q^{\langle\!\langle J,I\rangle\!\rangle}[\![I \cup M]\!][\![J \cup M]\!]$. $\square$

We are now ready for the first advertised proof of Theorem 1.

Proof of Theorem. Given $J, I \subseteq [n]$ with $d = |J| \leq |I| = e$, put $s = |J \cap I|$. After Lemma 2, we may assume $s = 0$. We proceed by induction on d, the base case being handled in Lemma 1.

Let j be the least element of J, i.e. $\ell(j|J^j) = 0$, and consider $(\mathcal{P}_{I,J\setminus j,j})$:

$$1 = \sum_{i\in I} r^{I\setminus i}_{ji} r^{J\setminus j}_{ij}\,.$$

In terms of quantum determinants, we have

$$[\![j|J^j]\!] = \sum_{i\in I} [\![j|I^i]\!][\![i|I^i]\!]^{-1}[\![i|J^j]\!]\,.$$

By induction, we may clear the denominator to the right and get

$$[\![j|J^j]\!][\![I]\!] = q^{\langle\!\langle J^j,I^i\rangle\!\rangle}\sum_{i\in I}(-q)^{\ell(i|I^i)}[\![j|I^i]\!][\![i|J^j]\!]\,. \tag{6}$$

On the otherhand, we may clear the denominator on the left at the expense of $q^{-\langle\!\langle j,i\rangle\!\rangle}$:

$$[\![I]\!][\![j|J^j]\!] = q^{-\langle\!\langle j,i\rangle\!\rangle}\sum_{i\in I}(-q)^{\ell(i|I^i)}[\![j|I^i]\!][\![i|J^j]\!]\,. \tag{7}$$

We are nearly done. First observe the following three facts.

$$q^{\langle\!\langle J^j,I^i\rangle\!\rangle} = q^{\langle\!\langle J^j,I\rangle\!\rangle} \qquad q^{-\langle\!\langle j,i\rangle\!\rangle} = q^{-\langle\!\langle j,I\rangle\!\rangle} \qquad q^{\langle\!\langle J,I\rangle\!\rangle} = q^{\langle\!\langle j,I\rangle\!\rangle}q^{\langle\!\langle J^j,I\rangle\!\rangle}$$

Using these observations to compare (6) and (7) finishes the proof. □

4. Preliminaries for $\mathcal{G}$-Proof

4.1. **Quantum flag algebra.**

The algebra $\mathcal{F}\ell_q(n)$ as presented below first appeared in [8].

Definition 5 (Quantum Flag Algebra)**.** The quantum flag algebra $\mathcal{F}\ell_q(n)$ is the $\Bbbk$-algebra generated by symbols $\{f_I : I \in [n]^d,\, 1 \le d \le n\}$ subject to the relations indicated below.

- *Alternating* relations $(\mathcal{A}_I)$: For any $I \in [n]^d$ and $\sigma \in \mathfrak{S}_d$,

$$f_I = \begin{cases} 0 & \text{if } I \text{ contains repeated indices} \\ (-q)^{-\ell(\sigma)}f_{\sigma I} & \text{if } \sigma I = (i_1 < i_2 < \cdots < i_d) \end{cases} \tag{8}$$

- *Young symmetry* relations $(\mathcal{Y}_{I,J})_{(a)}$: Fix $1 \le a \le d \le e \le n-a$. For any $I \in \binom{[n]}{e+a}$ and $J \in [n]^{d-a}$,

$$0 = \sum_{\Lambda\subseteq I, |\Lambda|=a}(-q)^{-\ell(I\setminus\Lambda|\Lambda)}f_{I\setminus\Lambda}f_{\Lambda|J} \tag{9}$$

- *Monomial straightening* relations $(\mathcal{M}_{J,I})$: For any $J, I \subseteq [n]$ with $|J| \le |I|$,

$$f_Jf_I = \sum_{\Lambda\subseteq I, |\Lambda|=|J|}(-q)^{\ell(\Lambda|I\setminus\Lambda)}f_{J|I\setminus\Lambda}f_\Lambda \tag{10}$$

Remark 3. Technically, we should have taken I, J to be tuples instead of sets in (9) and (10). Identify, e.g. $I = \{i_1 < i_2 < \cdots < i_d\}$ with $(i_1, i_2, \ldots, i_d)$. This abuse of notation will reoccur without further ado.

In their article, Taft and Towber construct an algebra map $\phi : \mathcal{F}\ell_q(n) \to \mathrm{M}_q(n)$ taking f_I to $[\![I]\!]$ and show that ϕ is monic, with image the subalgebra of $\mathrm{M}_q(n)$ generated by the quantum minors $\big\{[\![I]\!] : I \in [n]^d,\, 1 \leq d \leq n\big\}$.

We have already seen that the minors $[\![I]\!]$ often q-commute. This relation does not appear above, and so must be a consequence of relations (8)–(10). Abbreviate the right-hand side of (9) by $Y_{I,J;(a)}$. Also, we abbreviate the difference $(lhs - rhs)$ in (10) by $M_{J,I}$, and the difference $(lhs - rhs)$ in (1) by $C_{J,I}$ (replacing $[\![\text{-}]\!]$ by $f_{\text{-}}$). As (1),(9),(10) are all homogeneous, a likely guess is that $C_{J,I}$ is some $\Bbbk$-linear combination of a certain number of expressions $M_{K,L}$ and $Y_{M,N;(a)}$ (modulo the alternating relations). As illustrated in the example below, this simple guess works.

Example ($\{1\} \curvearrowright \{2,3,4\}$). We calculate the expressions $C_{1,234}$, $M_{1,234}$, and $Y_{1234,\emptyset;(1)}$ and arrange them as rows in Table 1. Viewing the table column by column, deduce $C_{1,234} = M_{1,234} + q^2 Y_{1234,\emptyset;(1)}$.

$C_{1,234}$	$f_1 f_{234}$				$-q^{-1} f_{234} f_1$
$M_{1,234}$	$f_1 f_{234}$	$-q^2 f_{123} f_4$	$+q^1 f_{124} f_3$	$-q^0 f_{134} f_2$	
$Y_{1234,\emptyset;(1)}$		$f_{123} f_4$	$-q^{-1} f_{124} f_3$	$+q^{-2} f_{134} f_2$	$-q^{-3} f_{234} f_1$

TABLE 1. Finding the relation $f_1 f_{234} - q^{-1} f_{234} f_1 = 0$.

While the proof idea will be simple ("perform Gaussian elimination"), the proof itself is not. We separate out the more interesting steps below.

4.2. **POset paths.** Given a set X, the elements of the power set $\mathcal{P}X$ have a partial ordering: for $A, B \in \mathcal{P}X$, we say $A < B$ if $A \subsetneq B$. We are interested in the case $X \subseteq [n]$ and we think of this POset as an edge-weighted, directed graph as follows.

Definition 6. Given $I, J \subseteq [n]$ such that $J \curvearrowright I$, the graph $\Gamma(J; I)$ has vertex set $\mathcal{V} = \mathcal{P}J$ and edge set $\mathcal{E} = \{(A,B) \mid A, B \in \mathcal{V},\, A \subsetneq B\}$. Each edge (A,B) of Γ has a weight α_A^B given by the function $\alpha : \mathcal{E} \to \Bbbk$,

$$\text{(11)} \quad \forall (A,B) \in \mathcal{E} : \quad \alpha_A^B = (-q)^{-\ell(J\setminus B|B\setminus A) - \ell(B\setminus A|A) + \big(2|J\setminus B| - |I|\big)|(B\setminus A)\cap J'|}$$

for J' is as in Definition 1.

Example. If $|J| = m$, then $\Gamma(J)$ has 2^m vertices and $\sum_{k=1}^m \binom{m}{k}(2^m - 1)$ edges. In Figure 1, we give an illustration of $\Gamma(\{1,5,6\})$, omitting two edges and many edge weights for legibility.

For the remainder of the subsection, we assume $J \cap I = \emptyset$. Write $J = J' \dot\cup J'' = \{j_1 < \ldots < j_{r'}\} \cup \{j_{r'+1} < \cdots < j_{r'+r''}\}$; also, put $|J| = r' + r'' = r$, $|I| = s$, and $s - r = t$.

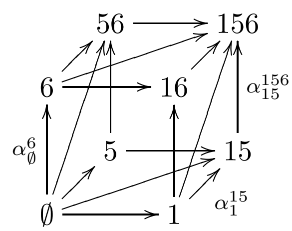

FIGURE 1. The graph $\Gamma(\{1,5,6\})$ (partially rendered).

In the graph $\Gamma(J;I)$, we consider *paths* and *path weights* defined as follows:

$$\mathfrak{P}_0 = \Big\{(A_1, A_2, \ldots, A_p) \mid A_i \subseteq J \text{ s.t. } \emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p \subsetneq J\Big\}$$

and $\mathfrak{P} = \mathfrak{P}_0 \cup \hat{0} \cup \hat{1}$, where $\hat{0} = (\emptyset)$, and

$$\hat{1} = (\{j_{r'+1}\}, \{j_{r'+1}, j_{r'+2}\}, \ldots, J'', \{j_{r'}, \ldots, j_r\}, \ldots, \{j_2, \ldots, j_r\}, J).$$

The *weight* $\alpha(\pi)$ of a path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0$ is the product of edge weights of the augmented path $(\emptyset, \pi, J)$:

$$\alpha_{\emptyset}^{A_1} \cdot \alpha_{A_1}^{A_2} \cdots \alpha_{A_{p-1}}^{A_p} \cdot \alpha_{A_p}^{J}.$$

We extend the definition of α to all of $\mathfrak{P}$ as follows. Notice that if $B = A$ in (11), we get $\alpha_A^A = 1$. With this broader definition of the weight function α, we may define $\alpha(\pi) = \alpha(\emptyset, \pi, J)$ for $\pi = \hat{0}, \hat{1}$ as well. Writing $\hat{1} = (A_1, \ldots, A_{r=|J|})$, the path $(A_1, \ldots, A_{r-1}) \in \mathfrak{P}_0$ will also be important. We label this special path $\pi^{\hat{1}}$.

Definition 7. Given a subset $K \subseteq J$, define $\mathsf{mM}(K)$ as follows. If $K \cap J' \neq \emptyset$, put $\mathsf{mM}(K) = \min(K \cap J')$. Otherwise, put $\mathsf{mM}(K) = \max(K \cap J'')$.

For any path $\pi = (A_1, \ldots, A_p)$, put $A_0 = \emptyset$ and $A_{p+1} = J$. Notice that $\hat{1}$ has the property that $A_k \setminus A_{k-1} \neq \mathsf{mM}(A_{k+1} \setminus A_{k-1})$ for all $1 \leq k < r$, but $A_r = \mathsf{mM}(A_{r+1} \setminus A_{r-1})$.

Definition 8. Fix a length $1 \leq p \leq r-1$. A path $(A_1, \ldots, A_p) \in \mathfrak{P}_0$ shall be called *regular* (or *regular at position* i_0), if $(\exists i_0)(1 \leq i_0 \leq p)$ satisfying: (a) $|A_i| = i\,(\forall\, 1 \leq i \leq i_0)$; (b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1})$ (again, taking $A_0 = \emptyset$ and $A_{p+1} = J$ if necessary). A sequence is called *irregular* if it is nowhere regular. Extend the notion of regularity to $\mathfrak{P}$ by calling $\hat{0}$ irregular and $\hat{1}$ regular.

Remark 4. The set $\mathfrak{P}$ is the disjoint union of its regular and irregular paths. We point out this tautology only to emphasize its importance in the coming proposition. Write $\mathfrak{P}'$ for the irregular paths, and $\mathfrak{P}''$ for the regular paths.

Proposition 7. *The subsets $\mathfrak{P}'$ and $\mathfrak{P}''$ of $\mathfrak{P}$ are equinumerous.*

We will build a bijective map $\wp$ between the two sets. Given an irregular path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0$, we insert a new set B so that $\wp(\pi)$ is regular at B:

(1) Find the unique i_0 satisfying: $(|A_i| = i \quad \forall i \leq i_0) \wedge (|A_{i_0+1}| > i_0+1)$.
(2) Compute $b = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0})$
(3) Put $B = A_{i_0} \cup \{b\}$.
(4) Define $\wp(\pi) := (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A_p)$.

For the remaining irregular path $\hat{0}$, we put $\wp(\hat{0}) = (\{j_1\})$, which agrees with the general definition of $\wp$ if we think of $\hat{0}$ as the empty path () instead of the path consisting of the empty set.

Example. Table 2 illustrates the action of $\wp$ on $\mathfrak{P}$ when $J = \{1, 5, 6\}$.

π	$\hat{0}$	(5)	(6)	(15)	(16)	(56)	$(5, 56)$
$\wp(\pi)$	(1)	$(5, 15)$	$(6, 16)$	$(1, 15)$	$(1, 16)$	$(6, 56)$	$\hat{1}$

TABLE 2. The pairing of $\mathfrak{P}'$ and $\mathfrak{P}''$ via $\wp$.

Proof of Proposition. We reach a proof in three steps.
Claim 1: $\wp(\mathfrak{P}') \subseteq \mathfrak{P}''$.

Take a path $\pi \in \mathfrak{P}'$ (i.e. a path with no regular points). The effect of $\wp$ is to insert a regular point at position $i_0 + 1$ (the spot where B sits), so the claim is proven if we can show $\wp(\pi) \in \mathfrak{P}$.

As $\wp(\hat{0})$ clearly belongs to $\mathfrak{P}$, we may focus on those $\pi \in \mathfrak{P}_0$. Also, it is plain to see that $\pi^{\hat{1}}$ is irregular, and $\wp(\pi^{\hat{1}}) = \hat{1}$. If $\wp$ is to be a bijection, we are left with the task of showing that $\wp(\mathfrak{P}' \cap \mathfrak{P}_0 \setminus \pi^{\hat{1}}) \subseteq \mathfrak{P}_0$

When $|A_p| < r - 1$, any B that is inserted will result in another path in $\mathfrak{P}_0$ (because $|B|$ must be less than r). When $|A_p| = r - 1$, there is some concern that we will have to insert a B at the end of the path, resulting in J being the new terminal vertex—disallowed in $\mathfrak{P}_0$. This cannot happen:

Case $p < r - 1$: At some point $1 \leq i_0 < p$, there is a jump in set-size greater than one when moving from A_{i_0} to A_{i_0+1}. Hence, the B to be inserted will not come at the end, but rather immediately after A_{i_0} to A_{i_0+1}

Case $p = r - 1$: The only path $(A_1, A_2, \ldots, A_{r-1}) \in \mathfrak{P}_0$ which is nowhere regular is the path $\pi^{\hat{1}}$.

Claim 2: $\wp$ is 1-1.

Suppose $\wp(A_1, \ldots, A_p) = \wp(A'_1, \ldots, A'_{p'})$, and suppose we insert B and B' respectively. By the nature of $\wp$, we have $p = p'$ and $i_0 \neq i'_0$. Take $i_0 < i'_0$. Also notice that $(A'_1, \ldots, A'_{p'}) = (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A'_{i'_0}, \ldots A'_{p'})$ In particular, B is a regular point of $(A'_1, \ldots, A'_{p'})$, and consequently, $(A'_1, \ldots, A'_{p'}) \notin \mathfrak{P}'$.

Claim 3: $\wp$ is onto.

Consider a path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}''$. If $p = 1$, then it is plain to see that the only irregular path is $\pi = (\{j_1\})$, which is the image of $(\emptyset)$ under $\wp$. So we consider $\pi \in \mathfrak{P}''$ with $p > 1$. Note that $|A_1| = 1$, for otherwise π cannot have any regular points. Now, locate the first $1 \leq i_0 \leq p$ with (a) $|A_{i_0}| = i_0$; and (b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1}$. The path $\pi' = (A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k)$ is in $\mathfrak{P}'$ and moreover, $\wp(\pi') = \pi$. □

Certainly one could cook up other bijections between the regular and irregular paths in $\mathfrak{P}$. The map we have used has an additional nice property.

Proposition 8. *The bijection $\wp$ from the proof of Proposition 7 is path-weight preserving.*

The result rests on

Lemma 3. Let $\emptyset \subseteq A \subseteq B \subseteq C \subseteq J$. Writing $\hat{B} = B \setminus A$ and $\hat{C} = C \setminus B$, we have

$$\alpha_A^B \alpha_B^C = \left[(-q)^{2\ell(B' \cap J' | C') - 2\ell(C' | B' \cap J'')} \right] \alpha_A^C . \tag{12}$$

Proof. From the definition of α_B^C, we have

$$\begin{aligned} \alpha_A^B &= (-q)^{-\ell(J \setminus B | \hat{B}) - \ell(\hat{B}|A) + \left(2|J \setminus B| - |I|\right)|\hat{B} \cap J'|} \\ \alpha_B^C &= (-q)^{-\ell(J \setminus C | \hat{C}) - \ell(\hat{C}|B) + \left(2|J \setminus C| - |I|\right)|\hat{C} \cap J'|} \\ \alpha_A^C &= (-q)^{-\ell(J \setminus C | \hat{B} \cup \hat{C}) - \ell(\hat{B} \cup \hat{C}|A) + \left(2|J \setminus C| - |I|\right)|(\hat{B} \cup \hat{C}) \cap J'|} \end{aligned}$$

Let us compare the exponents of α_A^C and $\alpha_A^B \alpha_B^C$:

$$\begin{aligned} \exp(\alpha_A^C) &= -\ell(J \setminus C | \hat{B}) - \ell(J \setminus C | \hat{C}) - \ell(\hat{C}|A) - \ell(\hat{B}|A) + \\ &\quad \left(2|J \setminus A| - 2|\hat{C}| - 2|\hat{B}| - |I|\right)\left(|\hat{B} \cap J'| + |\hat{C} \cap J'|\right), \end{aligned} \tag{13}$$

while

$$\begin{aligned} \exp(\alpha_A^B \alpha_B^C) &= -\ell(J \setminus B | \hat{B}) - \ell(J \setminus C | \hat{C}) - \ell(\hat{B}|A) - \ell(\hat{C}|B) + \\ &\quad \left(2|J \setminus B| - |I|\right)|\hat{B} \cap J'| + \left(2|J \setminus C| - |I|\right)|\hat{C} \cap J'| \\ &= -\left\{ \ell(J \setminus C | \hat{B}) + \ell(\hat{C}|\hat{B}) \right\} - \ell(J \setminus C | \hat{C}) - \ell(\hat{B}|A) - \\ &\quad \left\{ \ell(\hat{C}|A) + \ell(\hat{C}|\hat{B}) \right\} + \left\{ 2|J \setminus A| - 2|\hat{B}| - |I| \right\} |\hat{B} \cap J'| + \\ &\quad \left\{ 2|J \setminus A| - 2|\hat{B}| - 2|\hat{C}| - |I| \right\} |\hat{C} \cap J'| \\ &= 2|\hat{C}||\hat{B} \cap J'| - 2\ell(\hat{C}|\hat{B}) + \left\{ \exp(\alpha_A^C) \right\}. \end{aligned} \tag{14}$$

Notice that $2|\hat{C}||\hat{B} \cap J'| = 2\ell(\hat{C}|\hat{B} \cap J') + 2\ell(\hat{B} \cap J'|\hat{C})$, and that $-2\ell(\hat{C}|\hat{B}) = -2\ell(\hat{C}|\hat{B} \cap J') - 2\ell(\hat{C}|\hat{B} \cap J'')$. The discrepancy between (13) and (14) becomes $2\ell(\hat{B} \cap J'|\hat{C}) - 2\ell(\hat{C}|\hat{B} \cap J'')$, as desired. □

Now the proposition follows by comparing $\alpha(A_{i_0}, A_{i_0+1})$ and $\alpha(A_{i_0}, B, A_{i_0+1})$.

Proof of Proposition. Suppose that $\pi = (\ldots, A, C, \ldots)$, and that $\wp(\pi)$ inserts B immediately after A. Then $B = A \cup \mathsf{mM}(C \setminus A)$. Writing $b = \mathsf{mM}(C \setminus A)$, (12) implies

$$\alpha(\wp(\pi)) = \left[(-q)^{2\ell(b\cap J'|\hat{C}) - 2\ell(\hat{C}|b\cap J'')}\right] \cdot \alpha(\pi)\,.$$

Now, if $b \cap J' \neq \emptyset$, then b is the smallest element in $C \setminus A$, and in particular, $\ell(b|\hat{C}) = 0$. In this same case, $b \cap J'' = \emptyset$, so $\ell(\hat{C}|b \cap J'') = 0$ too. An analogous argument works for the case $b \cap J' = \emptyset$. □

One more interesting fact about $\Gamma(J;I)$ and $\mathfrak{P}$ is worth mentioning. When calculating $\alpha(\pi^{\hat{1}})$ using (12), the twos introduced in the exponents there all disappear.

Proposition 9. *Given, J, J', J'', and $\pi^{\hat{1}}$ as above, we have*

$$\alpha(\pi^{\hat{1}}) = (-q)^{|J'|(|J'|-1) - |J''|(|J''|-1)} \times \alpha^J_\emptyset\,. \tag{15}$$

Proof. Applying (12) repeatedly to the expression $\alpha(\pi^{\hat{1}})$ we see that

$$\begin{aligned}
\alpha(\pi^{\hat{1}}) &= \left[(-q)^{2\ell(j_{r'+1}\cap J'|j_{r'+2}) - 2\ell(j_{r'+2}|j_{r'+1}\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'+1}j_{r'+2}} \alpha_{j_{r'+1}j_{r'+2}}^{j_{r'+1}j_{r'+2}j_{r'+3}} \cdots \alpha^J_{j_2\cdots j_r} \\
&= (-q)^{-2(1)} \left[(-q)^{2\ell(j_{r'+2}\cap J'|j_{r'+3}) - 2\ell(j_{r'+3}|j_{r'+2}\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'+1}j_{r'+2}j_{r'+3}} \cdots \alpha^J_{j_2\cdots j_r} \\
&= (-q)^{-2(1)-2(2)} \left[(-q)^{2\ell(j_{r'+3}\cap J'|j_{r'+4}) - 2\ell(j_{r'+4}|j_{r'+3}\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'+1}j_{r'+2}j_{r'+3}j_{r'+4}} \cdots \alpha^J_{j_2\cdots j_r} \\
&\ \ \vdots \\
&= (-q)^{-2(1)-\cdots-2(|J''|-1)} \left[(-q)^{2\ell(j_r\cap J'|j_{r'}) - 2\ell(j_{r'}|j_r\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'}\cdots j_r} \cdots \alpha^J_{j_2\cdots j_r} \\
&= (-q)^{-2\frac{(|J''|-1)|J''|}{2}} (-q)^{0-0} \left[(-q)^{2\ell(j_{r'}\cap J'|j_{r'-1}) - 2\ell(j_{r'-1}|j_{r'}\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'-1}\cdots j_r} \cdots \alpha^J_{j_2\cdots j_r} \\
&= (-q)^{2(1)} (-q)^{-|J''|(|J''|-1)} \left[(-q)^{2\ell(j_{r'-1}\cap J'|j_{r'-2}) - 2\ell(j_{r'-2}|j_{r'-1}\cap J'')}\right] \times \\
&\quad \alpha_\emptyset^{j_{r'-2}\cdots j_r} \cdots \alpha^J_{j_2\cdots j_r} \\
&\ \ \vdots \\
&= (-q)^{2(1)+\cdots+2(|J'|-1)} (-q)^{-|J''|(|J''|-1)} \times \alpha^J_\emptyset \\
&= (-q)^{|J'|(|J'|-1) - |J''|(|J''|-1)} \times \alpha^J_\emptyset\,. \quad \square
\end{aligned}$$

5. $\mathcal{G}$-Proof of Theorem

We keep the notations $J', J'', r', r'', r, s, t$ from Section 4.2, and as we did there, we only consider the case $J \cap I = \emptyset$.[2] Before we dive in, we define a new quantity $CM_{J,I}(\theta)$.

$$\begin{aligned} C_{J,I} - M_{J,I} &= -q^{|J''|-|J'|} f_I f_J + \left(\sum_{\Lambda \subseteq I,\, |\Lambda| = r} (-q)^{\ell(\Lambda|I^\Lambda)} f_{J|I\setminus\Lambda} f_\Lambda \right) \\ &= \sum_{\Lambda\subseteq I} (-q)^{|J'|t} (-q)^{-\ell(I^\Lambda|\Lambda)} f_{J\cup(I\setminus\Lambda)} f_\Lambda - q^{|J''|-|J'|} f_I f_J \\ &= \sum_{\Lambda\subseteq I} (-q)^{|J'|t+|J''||J|} (-q)^{-\ell((J\cup I)^\Lambda|\Lambda)} f_{(J\cup I)\setminus\Lambda} f_\Lambda - q^{|J''|-|J'|} f_I f_J \end{aligned}$$

Here, we have replaced $\ell(\Lambda|I^\Lambda)$ with $|I^\Lambda||\Lambda| - \ell(I^\Lambda|\Lambda)$ and $\ell(J|I^\Lambda)$ with $|J||I^\Lambda| - \ell(I^\Lambda|J)$.

$$CM_{J,I}(\theta) := (-q)^{|J'|t+|J''||J|} \left(\sum_{\Lambda\subseteq I} (-q)^{-\ell((J\cup I)^\Lambda|\Lambda)} f_{(J\cup I)\setminus\Lambda} f_\Lambda - \theta f_I f_J \right).$$

We prove the theorem in steps:

Proposition 10. *Suppose $I, J \subseteq [n]$ are such that $J \curvearrowright I$. With $CM_{J,I}(\theta)$ and $Y_{L,K;(a)}$ as defined above,*

$$\sum_{\emptyset\subseteq K\subsetneq J} \eta_K \cdot Y_{(I\cup J)\setminus K, K; (r-|K|)} = CM_{J,I}(\theta)$$

for some constants $\{\eta_K \in Z[q, q^{-1}] : \emptyset \subseteq K \subsetneq J\}$ and $\theta \in Z[q, q^{-1}]$.

Proposition 11. *In the notation above, $\theta = (-q)^{-|J'|t-|J''||J|} q^{|J''|-|J'|}$.*

The alternating property of the symbols f_K and the product in $\mathcal{F}\ell_q(n)$ play no role in our proof, so we begin by eliminating these distractions. Let V be the vector space over $\Bbbk$ with basis $\{e_{(A,B)} : A \cup B = I \cup J, A \cap B = \emptyset, \text{ and } |B| = r\}$. There is a $\Bbbk$-linear map $\mu : V \to \mathcal{F}\ell_q(n)$, sending $e_{A,B}$ to $f_A f_B$. The vectors

$$v^\theta := \sum_{\Lambda\subseteq I} (-q)^{-\ell((I\cup J)^\Lambda|\Lambda)} e_{(I\cup J)\setminus\Lambda,\Lambda} - \theta e_{I,J}$$

and (for each $\emptyset \subseteq K \subsetneq J$)

$$v^K := \sum_{\Lambda\subseteq(I\cup J), |\Lambda| = r-|K|} (-q)^{-\ell((I\cup J^K)^\Lambda|\Lambda)} (-q)^{-\ell(\Lambda|K)} e_{(I\cup J)\setminus(K\cup\Lambda), K\cup\Lambda}$$

[2]Only minor changes to this proof are needed to prove the theorem in the general setting (e.g. replacing every instance of J below with $J_0 := J \setminus I$). In the interest of avoiding even more notation, we leave this work to the reader.

have familiar images. Check that $\mu((-q)^{|J'|t+|J''||J|}\cdot v^\theta) = CM_{J,I}(\theta)$ and $\mu(v^K) = Y_{(I\cup J)\setminus K,K;(r-|K|)}$.

Proposition 10 will be proven if we can show that v^θ is a linear combination of the v^K for some θ. This is not immediate as the span of the vectors v^K has dimension (at most, *a priori*) 2^r-1, while V is $\binom{r+s}{r}$ dimensional.

Definition 9. For each $K\in\mathcal{P}J$, let $V_{(K)} = \operatorname{span}_{\Bbbk}\big\{e_{A,B} : B\cap J = K\big\}$. Clearly, V is graded by the POset $\mathcal{P}J$, i.e., $V = \bigoplus_{K\in\mathcal{P}J} V_{(K)}$. For each $K\in\mathcal{P}J$, define the distinguished element e^K by

$$e^K = \sum_{\Lambda\subseteq I, |\Lambda|=r-|K|} (-q)^{-\ell((I\cup J)^{(K\cup\Lambda)}|\Lambda)}(-q)^{-\ell(\Lambda|K)} e_{(I\cup J)\setminus(\Lambda\cup K),\Lambda\cup K}.$$

For any $v\in V$, write $(v)_{(K)}$ for the component of v in $V_{(K)}$, that is, $v = \sum_K (v)_{(K)}$.

Notice that $e^J = e_{I,J}$, and that

$$e^\emptyset = \sum_{\Lambda\subseteq I, |\Lambda|=r} (-q)^{-\ell((I\cup J)^\Lambda|\Lambda)} e_{(I\cup J)\setminus\Lambda,\Lambda}$$

In other words, $v^\theta = e^\emptyset - \theta e^J$. Good fortune provides that the $v^{K'}$ may also be expressed in terms of the e^K.

Lemma 4. For each $K'\in\mathcal{P}J\setminus J$, there are constants $\alpha^K_{K'}\in\Bbbk$ satisfying

$$v^{K'} = \sum_{K\in\mathcal{P}J}\alpha^K_{K'} e^K\,.$$

Remark 5. As the proof will show, these $\alpha^K_{K'}$ are precisely the edge-weights of $\Gamma(J;I)$ from Section 4.2, in particular $\alpha^K_K = 1$. It will also show that $\alpha^K_{K'} = 0$ if $K'\not\leq K$ in the POset $\mathcal{P}J$, a critical ingredient in the approaching Gaussian elimination argument.

Proof of Lemma. Fixing a subset K', if $K\supsetneq K'$, we write $\hat{K} = K\setminus K'$. Similarly, let $\hat\Lambda = \Lambda\setminus J$. Studying $v^{K'}$, we see that

$$\begin{aligned}
v^{K'} &= \sum_{\substack{\Lambda\subseteq(I\cup J)\setminus K'\\ |\Lambda|=r-|K'|}} (-q)^{-\ell((I\cup J^{K'})^\Lambda|\Lambda)}(-q)^{-\ell(\Lambda|K')} e_{(I\cup J)\setminus(\Lambda\cup K'),\Lambda\cup K'}\\
&= \sum_{K\in\mathcal{P}J} (v^{K'})_{(K)}\\
&= \sum_{K\in\mathcal{P}J}\ \sum_{\substack{\Lambda\subseteq(I\cup J)\setminus K'\\ \Lambda\cap J=\hat K}} (-q)^{-\ell((I\cup J)^{(\hat\Lambda\cup K)}|\hat\Lambda\cup\hat K)}\times\\
&\qquad (-q)^{-\ell(\hat\Lambda\cup\hat K|K')} e_{(I\cup J)\setminus(\hat\Lambda\cup K),\hat\Lambda\cup K}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{P}J} (-q)^{-\ell((I^{\hat\Lambda})\cup(J^K)|\hat K)}(-q)^{-\ell(\hat K|K')} \times \\
&\quad \left(\sum_{\substack{\hat\Lambda\subseteq I\\ |\hat\Lambda|=r-|K|}} (-q)^{-\ell((I\cup J)^{(\hat\Lambda\cup K)}|\hat\Lambda)}(-q)^{-\ell(\hat\Lambda|K')} e_{(I\cup J)\setminus(\hat\Lambda\cup K),\hat\Lambda\cup K} \right).
\end{aligned}
$$

Why can we perform this last step? Because $J \curvearrowright I$, the expression $\ell(I^{\hat\Lambda}|\hat K)$ does not actually depend on $\hat\Lambda$, only on $|\hat\Lambda|$. Indeed, it equals $|I\setminus\hat\Lambda|\cdot|\hat K\cap J'|$. Multiplying and dividing by $(-q)^{-\ell(\hat\Lambda|\hat K)}$, we rewrite this last expression as

$$
\begin{aligned}
v^{K'} &= \sum_K (-q)^{-\ell((I^{\hat\Lambda})\cup(J^K)|\hat K)}(-q)^{-\ell(\hat K|K')+\ell(\hat\Lambda|\hat K)} \times \\
&\quad \left(\sum_{\hat\Lambda\subseteq I, |\hat\Lambda|=r-|K|} (-q)^{-\ell((I\cup J)^{(\hat\Lambda\cup K)}|\hat\Lambda)}(-q)^{-\ell(\hat\Lambda|K)} e_{(I\cup J)\setminus(\hat\Lambda\cup K),\hat\Lambda\cup K} \right) \\
&= \sum_{K'\le K} (-q)^{\left(2|J\setminus K|-|I|\right)|\hat K\cap J'|-\ell(J^K|\hat K)-\ell(\hat K|K')} \times \left(e^K\right) \\
&= \sum_{K'\le K} \alpha^K_{K'} e^K. \quad \square
\end{aligned}
$$

Corollary 12. *For any $v^{K'}, v^K$ with $K' < K$ in the POset $\mathcal{P}J$, and for the same constants $\alpha^K_{K'}$ as defined above, we have*

$$(v^{K'} - \alpha^K_{K'} v^K)_{(K)} = 0.$$

Proof of Proposition 10. We use the corollary to perform a certain Gaussian elimination on the "matrix" of the vectors v^K. Table 3 displays this matrix for the POset $\mathcal{P}(\{1,5,6\})$ should make our intentions clear.

	$e^\emptyset$	e^1	e^5	e^6	e^{15}	e^{16}	e^{56}	e^{156}
v^{15}					1			α^{156}_{15}
v^{16}						1		α^{156}_{16}
v^{56}							1	α^{156}_{56}
v^1		1			α^{15}_1	α^{16}_1		α^{156}_1
v^5			1		α^{15}_5		α^{56}_5	α^{156}_5
v^6				1		α^{16}_6	α^{56}_6	α^{156}_6
$v^\emptyset$	1	$\alpha^1_\emptyset$	$\alpha^5_\emptyset$	$\alpha^6_\emptyset$	$\alpha^{15}_\emptyset$	$\alpha^{16}_\emptyset$	$\alpha^{56}_\emptyset$	$\alpha^{156}_\emptyset$

TABLE 3. Writing the vectors $v^{K'}$ in terms of the e^K.

Performing Gaussian elimination between the rows in the first two layers of the matrix, we see that the new rows in the second layer—who began their life with $|J|+1$ nonzero entries—now have exactly two nonzero entries.

$$\begin{aligned}(v^{J\setminus\{k,l\}})' &= v^{J\setminus\{k,l\}} - \alpha^{J\setminus k}_{J\setminus\{k,l\}} v^{J\setminus k} - \alpha^{J\setminus l}_{J\setminus\{k,l\}} v^{J\setminus l} \\ &= e^{J\setminus\{k,l\}} + \left(\alpha^{J}_{J\setminus\{k,l\}} - \alpha^{J\setminus k}_{J\setminus\{k,l\}}\alpha^{J}_{J\setminus k} - \alpha^{J\setminus l}_{J\setminus\{k,l\}}\alpha^{J}_{J\setminus l}\right) e^{J},\end{aligned}$$

e.g., v^1 from Table 3 becomes $(v^1)' = e^1 + \left(\alpha^{156}_1 - \alpha^{15}_1\alpha^{156}_{15} - \alpha^{16}_1\alpha^{156}_{16}\right) e^{156}$. Marching down the layers of this matrix one-by-one, we see that the new final row is given by $(v^{\emptyset})' = e^{\emptyset} + \theta e^{J} = v^{\theta}$ for some θ. □

Proof of Proposition 11. Careful bookkeeping shows that

$$\begin{aligned}\theta &= \alpha^{J}_{\emptyset} - \left(\sum_{\emptyset\subsetneq K\subsetneq J} \alpha^{K}_{\emptyset}\alpha^{J}_{K}\right) + \left(\sum_{\emptyset\subsetneq K_1\subsetneq K_2\subsetneq J} \alpha^{K_1}_{\emptyset}\alpha^{K_2}_{K_1}\alpha^{J}_{K_2}\right) - \cdots \\ &\quad \cdots + (-1)^{|J|-1}\left(\sum_{\emptyset\subsetneq K_1\subsetneq\cdots\subsetneq K_{|J|-1}\subsetneq J} \alpha^{K_1}_{\emptyset}\alpha^{K_2}_{K_1}\cdots\alpha^{J}_{K_{r-1}}\right). \qquad (16)\end{aligned}$$

In other words, θ is a signed sum of path weights $\alpha(\pi)$, π running over all paths in $\mathfrak{P}$ save for $\hat{1}$. As the sign attached to π is the same as the length of π, and as the bijection $\wp$ from Section 4.2 increases length by one but preserves path weight, we immediately conclude

$$\begin{aligned}\theta &= (-1)^{|J|-1}\alpha(\pi^{\hat{1}}) \\ &= (-1)^{|J|-1}(-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha^{J}_{\emptyset} \\ &= (-1)^{|J|-1}(-q)^{|J''|-|J'|}(-q)^{|J'||J'|-|J''||J''|-|I||J'|} \\ &= q^{|J''|-|J'|}(-q)^{|J'||J'|-|J''||J''|-|J''||J'|-|I||J'|-(|J'|+|J''|+t)|J'|+|J''||J'|} \\ &= q^{|J''|-|J'|}(-q)^{-|J'|t-|J''||J|}. \quad \square\end{aligned}$$

With Proposition 11 proven, Theorem 1 is finally demonstrated (modulo the Taft-Towber isomorphism ϕ). Moreover, we achieve the second goal stated in the introduction. A brief discussion of the first goal follows.

6. On Quantum- and Quasi- Flag Varieties

The algebra $\mathcal{F}\ell_q(n)$ is a quantum deformation of the classic multihomogeneous coordinate ring of the full flag variety over GL_n. The deformation was constructed in a somewhat ad-hoc manner, and we would like to know whether a theory of *noncommutative flag varieties* using quasideterminants could help explain the choices for the relations in $\mathcal{F}\ell_q(n)$. In [6], it is shown that any relation $(\mathcal{Y}_{I,J})_{(a)}$ has a quasi-Plücker coordinate origin. Section 3 shows that (1) does too. The second proof of Theorem 1 shows that a great many instances of $(\mathcal{M}_{J,I})$ do as well; to see this, note that the roles of $M_{J,I}$ and $C_{J,I}$ were interchangeable there. The question of whether and to what

extent the gap (case $J \not\curvearrowright I$) may be filled by finding new quasi-Plücker coordinate identities is an interesting one. Toward a partial answer, we leave the reader to verify that

$$(\mathcal{P}_{I,J^j,j}) \Rightarrow (\mathcal{M}_{J,I})$$

whenever $I, J \subseteq [n]$ are such that $|J| \leq |I|$ and $J \setminus j \subseteq I$.

References

[1] I. M. Gel′fand and V. S. Retakh. Determinants of matrices over noncommutative rings. *Funktsional. Anal. i Prilozhen.*, 25(2):13–25, 96, 1991.

[2] I. M. Gelfand and V. S. Retakh. Quasideterminants, I. *Selecta Math. (N.S.)*, 3(4):517–546, 1997.

[3] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson. Quasideterminants. *Adv. in Math.*, 193(1):56–141, 2005.

[4] A. C. Kelly, T. H. Lenagan, and L. Rigal. Ring theoretic properties of quantum Grassmannians. *J. Algebra Appl.*, 3(1):9–30, 2004.

[5] Daniel Krob and Bernard Leclerc. Minor identities for quasi-determinants and quantum determinants. *Comm. Math. Phys.*, 169(1):1–23, 1995.

[6] Aaron Lauve. Quantum- and quasi-Plücker coordinates. *J. Algebra*, 296(2):440–461.

[7] Bernard Leclerc and Andrei Zelevinsky. Quasicommuting families of quantum Plücker coordinates. In *Kirillov's seminar on representation theory*, volume 181 of *Amer. Math. Soc. Transl. Ser. 2*, pages 85–108. Amer. Math. Soc., Providence, RI, 1998.

[8] Earl Taft and Jacob Towber. Quantum deformation of flag schemes and Grassmann schemes, I. A q-deformation of the shape-algebra for $\mathrm{GL}(n)$. *J. Algebra*, 142(1):1–36, 1991.

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